$\equiv [(\sim p \wedge \sim q) \wedge r) \vee [\{(q \wedge r) \wedge r\} \vee (p \wedge r)]$

$\equiv [(\sim p \wedge \sim q) \wedge r) \vee [\{(q \wedge r) \vee (p \wedge r)\}]$

$\equiv [(\sim p \wedge \sim q) \wedge r) \vee [(q \wedge p) \vee r]$

$\equiv [(\sim p \wedge \sim q)] \vee [(q \vee p)] \vee r$

$\equiv [\sim (p \vee q) \vee (p \vee q)] \wedge r$

$\equiv r$ R.H.S.

Dual of the given statement can be obtained by interchanging $\wedge$ and $\vee$ and is $\sim p \vee (\sim q \vee r) \wedge (q \vee r) \wedge (p \vee r) \equiv r$.

Check Your Progress

1. Let P be "It is cold" and let q be "It is training". Give a simple verbal sentence which describes each of the following statements.
   (a) $\neg P$ (b) $s (s_1 + 1)$
   (c) $P \vee q$ (d) $q \vee \neg P$
2. Let P be "Ram reads Hindustan Times", let q be "Ram reads Times of India," and let r be "Ram reads NBT! write each of the following in symbolic form:
   (a) Ram reads Hindustan Times on Times of India not NBT.
   (b) Ram reads Hindustan Times and Times of India, on he does not read Hindustan Times and NBT.
   (c) It is not true that Ram reads Hindustan Times but not NBT.
   (d) It is not true that Ram reads NBT on Times of India but not Hindustan Times.
3. Determine the truth value of each of the following statements:
   (a) 4 + 2 = 5 and 6 + 3 = 9
   (b) 3 + 2 = 5 and 6 + 1 = 7
   (c) 4 + 5 = 9 and 1 + 2 = 4
   (d) 3 + 2 = 5 and 4 + 7 = 11
4. Find the truth table of $\neg P \wedge q$.
5. Verify that the propostion $P \vee \neg (P \vee q)$ is tautalogy.
6. Show that the propostions $\neg (P \wedge q)$ and $\neg P \vee \neg q$ are logically equivalent.
7. Rewrite the following statements without using the conditonal:
   (a) If it is cold, he means a hat.
   (b) If productivity increases, then wages rise.

Contd....

8. Determine the contraposition of each statement:
   (a) If John is a poet, then he is poor;
   (b) Only if he studies more then he will pass the test.
9. Show that the following arguments is a fallacy :

$$P \rightarrow q, \neg P + \neg q.$$

10. Write the negation of each statement as simply as possible.

$$P \rightarrow q, \neg q + \neg P.$$

11. Let A = {1, 2, 3, 4, 5}. Determine the truth table value of each of the following statements:
    (a) $(\exists x \in A)\ (x + 3 = 10)$
    (b) $(\forall x \in A)\ (x + 3 > 10)$
    (c) $(\exists x \in A)\ (x + 3 < 5)$
    (d) $(\forall x \in A)\ (x + 3 \leq 7)$
12. Determine the truth table value of each of the following statements where U = {1, 2, 3} is the universal set:
    (a) $\exists x\ \forall y, x^2 < y + 1;$
    (b) $\forall x\ \exists y, x^2 + y^2 < 12$
    (c) $\forall x \forall y, x^2 + y^2 < 12$
13. Negate each of the following statements:
    (a) $\exists x \forall y, P(x, y)$
    (b) $\exists x \forall y, P(x, y)$
    (c) $\exists x \exists y \exists z, P(x, y, z)$
14. Let P(x) denote the sentence "x + 2 > 5". State whether on not P(x) is a propositional function on each of the following sets:
    (a) N, the set of positive integers;
    (b) M = (– 1, – 2, – 3, ...)
    (c) C, the set of complex numbers.
15. Negative each of the following statements:
    (a) All students live in the domintonics
    (b) All Mathematics majors are values.
    (c) Some students are 25 (years) on older.

Contd....

16. Let P denote the is rich and let q denote "He is happy". Write each statement in symbolic form using P and q. Note that "He is poor" and he is unhappy" are equivalent to. $\neg P$ and $\neg q$ respectively.
    (a) If he is rich, then he is unhappy
    (b) He is neither rich nor happy
    (c) It is necessary to be poor in order to be happy.
    (d) To be poor is to be unhappy.
17. Find the truths tables for :
    (a) $P \vee \neg q$
    (b) $P \wedge \neg q$
18. Show that :
    (a) $P \wedge q$ logically implies $P \leftrightarrow q$
    (b) $P \rightarrow \neg q$ does not logically imply $P \leftrightarrow q$.
19. Let A = {1, 2, ..., 9, 10}. Consider each of the following sentences. If it is a statement, then determine its truth value. If it is a propositional function, determine its truth set.
    (a) $(\forall x \in A)\ (\forall y \in A)\ (x + y < 14)$
    (b) $(\forall y \in A)\ (x + y < 14)$
    (c) $(\forall x \in A)\ (\forall y \in A)\ (x + y < 14)$
    (d) $(\exists y \in A)\ (x + y < 14)$
20. Negative each of the following statement:
    (a) If the teacher is absent, then some students do not complete their homework.
    (b) All the students completed their homework and the teacher is present.
    (c) Some of the students did not compute their homework on teacher is absent.

1.10 LET US SUM UP

"A proposition is a statement that is either true or false but not both".

The statement "It is not the case that P" is another proposition, called the negation of P. The negation of P is denoted by GP.

The propositions "P or q" denoted by $P \vee q$, is the proposition that is false when P and q are both false and true otherwise the proposition $P \vee q$ is called the disjunction of P and q.

The proposition "P and 'V'" denoted by $P \wedge q$, is the proposition that is true when both P and q are true and is false otherwise and denoted by "$\wedge$". It is called conjunction.

The implication $P \rightarrow q$ is the proposition that is false, when P is true and q is false and true otherwise. In this implication P is called the hypothesis (or antecedent or premise) and q is called conclusion (or consequence).

A statement formula (expression involving prepositional variable) that is neither a tautology nor a contradiction is called a contingency.

Two statements A and B in variable P_1 ...... Pn ($n \geq U$) are said to be equivalent if they acquire the same truth values for all interpretation, i.e., they have identical truth values

A statement that is true for all possible values of its propositional variable is called a tautology. A statement that is always false is called a contradiction and a statement that can be either true or false depending on the truth values of its propositional variables is called a contingency.

1.11 KEYWORDS

Proposition: A proposition is a statement that is either true or false but not both

Truth Table: A truth table displays the relationship between the truth values of propositions

Tautology: A statement that is true for all possible values of its propositional variable is called a tautology.

Duality: Any two formulas A and A* are said to be duals of each other if one can be obtained from the other by replaying $\wedge$ by $\vee$ and $\vee$ by $\wedge$.

Quantifiers: When all the variables in a propositional function are assigned values, the resulting statement has a truth value.

1.12 QUESTIONS FOR DISCUSSION

1. Consider the conditional proposition $P \downarrow q$. The simple propositions $q \rightarrow P$, $\neg P \rightarrow \neg q$ and $\neg q \rightarrow \neg P$ are called respectively the converse inverse and contra positive of the conditional $P \rightarrow q$. Which if any of these proposition are logically equivalent to $P \rightarrow q$?
2. Write the negation of each statement as simple as possible.
   (a) If she works, she will down money.
   (b) He swims if and only if the water is warm.
   (c) If it shows, then they do not drive the can.
3. Verify that the proposition $(P \wedge q) \wedge \neg (P \vee q)$ is a contradiction.

Check Your Progress: Modal Answers

1. In each case, translate ∧, ∨ and ~ to read "and", "or", and "It is false that" on "hat", respectively and then simplify the English sentence.
   (a) It is not cold
   (b) It is cold and training
   (c) It is cold on it is training
   (d) It is training or it is cold.
2. Vise ∨ for "an", ∨ for "and", (or, its logical equivalent, "but"), and ¬ for "not" (negation).
   (a) $(P \vee q) \wedge \neg r$
   (b) $(P \wedge q) \vee \neg (P \wedge r)$
   (c) $\neg (P \wedge \neg r)$
   (d) $\neg [(r \vee q) \wedge \neg P]$
3. The statement "P and q" is true only when both sub statements are true. Thus
   (a) False
   (b) True
   (c) False
   (d) True
4.

P	q	¬P	¬P ∧ q
T	T	F	F
T	F	F	F
F	T	T	T
F	F	T	F

5. Construct the truth table of $P \vee \neg (P \wedge q)$. Since the truth values of $P \vee \neg (P \wedge q)$ is T for all values of P and q, the proposition is a tautology.

P	q	P ∨ q	¬(P ∧ q)	P ∨ ¬(P ∨ q)
T	T	T	F	T
T	F	F	T	T
F	T	F	T	T
F	F	F	T	T

6. Construct the truth tables for $\neg (P \wedge q)$ and $\neg P \vee \neg q$. Since the truth tables are the same, the propostions $\neg (P \wedge q)$ and $\neg P \wedge q$ are logically equivalent and we can write

   $\neg (P \wedge q) \equiv P \vee \neg q$

Contd...

P	q	P ∧ q	¬ (P ∧ q)	P	q	¬ P	q	¬ P ∧ ¬ q
T	T	T	F	T	T	F	F	F
T	F	F	T	T	F	F	T	T
F	T	F	T	F	T	T	F	T
F	F	F	T	F	F	T	T	T

7. Recall that "If P then q" is equivalent to "Not P or q"; that is, P → q ≡ ¬P ∨ q. Hence,

   (a) It is not cold on he means a hot.

   (b) Productivity does not increase on wages rise.

8. (a) The contrapositive of P → q is ¬ q → ¬ P. Hence the contrapositive of the given statement is

   If John is not poor, then he is not a poet.

   (b) The given statement is equivalent to "If Marc Passes the test, then he studied." Hence its contrapostive is

   If Marc does not study, then he will not pass the test.

9. Construct the truth table for [(P → q) ∧ ¬P] → ¬q.

   Since the proposition [(P → q) ∧ ¬ P] → ¬ q is not a tautology, the argument is a fallacy. Equivalently, the argument is a fallacy since in third line of the truth table P → q and ¬ P are true but ¬ q is false.

P	*q*	¬P	¬ *q*	Conditional P → *q*	Converse *q* → P	Inverse ¬ P → ¬ *q*	Contrapositive
T	T	F	F	T	T	T	T
T	F	F	T	F	T	T	F
F	T	T	F	T	F	F	T
F	F	T	T	T	T	T	T

10. Construct the truth table for [(P → q) ∧ ¬ q] → P. Since the proposition [(P → q)∧ ¬ q] → ¬ P is a tautology, the argument is valid..

P	*q*	(P → *q*) ∧ ¬ *q*] →			¬P					
T	T	T	T	T	F	F	T	T	F	T
T	F	T	F	F	F	T	F	T	F	T
F	T	F	T	T	F	F	T	T	T	F
F	F	F	T	F	T	T	F	T	T	F
Step		1	2	1	3	2	1	4	2	

Contd...

11. Construct the truth tables of the premises and conclusion as shown below. Now, P → ¬ q, r → q, and r and true simultaneously only in the fifth now of the table, where P is also true. Hence, the argument is valid.

	P	q	r	P → q	r → q	q
1	T	T	T	F	T	F
2	T	T	F	F	T	F
3	T	F	T	T	F	F
4	T	F	F	T	T	F
5	F	T	T	T	T	T
6	F	T	F	T	T	T
7	F	F	T	T	T	T
8	F	F	F	T	T	T

12. (a) False. For no number in A is a solution to x + 3 = 10.
 (b) True. For every number in A satisfies x + 3 < 10.
 (c) True. For if x_0 = 1, then x_0 + 3 <, i.e., 1 is a solution.
 (d) False. For if x_0 = 5, then x_0 + 3 is not less than on equal.
 (e) In other words, 5 is not a solution to the given condition.
13. (a) True. For if x = 1, then 1, 2 and 3 are all solutions to 1 < y + 1.
 (b) True. For each x_0, let y = 1; then $x_0^2 + 12$ is a true statement.
 (c) False. For if $x_0 = 2$ and $y_0 = 3$, then $x_0^2 + y_0^2 < 12$ is not a true statement.
14. Use ¬ ∀x P(x) ≡ ∃x / P(x) and ¬ ∃x P(x) ≡ ∀x P(x);
 (a) ¬ (∃x, ∀y, P(x, y)) ≡ ∀x, ∃y, ¬ P(x, y)
 (b) ¬ (∀x, ∀y, P(x, y)) ≡ ∃x, ∃y, ¬ P(x, y)
 (c) ¬ (∃y∃x ∀z, P(x, y, z)) ≡ ∀y, ∀x∃z, ¬ P(x, y, z)
15. (a) Yes.
 (b) Although P(x) is false for every element in M, P(x) is. still al propostional function on M.
 (c) No. Note that 2i + 2 > 5 does not have meaning in other words in equalities are not defined for complex number.
16. (a) At least one student does not live in the donmitonic.
 (b) At least one Mathematics major is female.
 (c) None of the students is 25 on older.

Contd...

17. (a) P

18. (a)

P	q	~q	P ∨ ~q
T	T	F	T
T	F	T	T
F	T	F	F
F	F	T	T

(b)

P	q	¬P	¬q	¬P ∨ ¬q
T	T	F	F	F
T	F	F	T	F
F	T	T	F	F
F	F	T	T	T

19. (a) The open sentence in two variables is preceded by two quantifiers; hence it is a statement. Moreover, the statement is true.

    (b) The open sentence is preceded by one quantifier; hence it is a propositional function of the other variable. Note that for every $y \in A$, $x_0 + y < 14$ if and only if $x_0 = 1, 2,$ or 3. Hence the truth set is $\{1, 2, 3\}$.

    (c) It is a statement and it is false: if $x_0 = 8$ and $y_0 = 9$, then $x_0 + y_0 < 14$ is not true.

    (d) It is an open sentence in x. The truth set is A itself.

20. (a) The teacher is absent and all the students completed their homework.

    (b) Some of the students did not complete their homework and the teacher is absent.

    (c) All the students completed their homework and the teacher is present.

1.13 SUGGESTED READINGS

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UNIT II

LESSON

2

SET THEORY

CONTENTS

2.0 AIMS AND OBJECTIVES

After studying this lesson, you will be able to:

- Understand the various ways to represent the sets
- Understand the infinite sets
- Understand the set operations
- Understand the union and intersection terminology

- Explain the symmetric difference
- Explain the law's of identities
- Gain knowledge about the symbolic logic

2.1 INTRODUCTION

A set is a collection of definite distinguishable objects such that, given a set and an object, we can ascertain whether or not the specified object is included in the set.

Sets A set is a well-defined collection of distinct objects.

By a 'well defined' collection of objects, we mean that there is a rule (s) by means of which it is possible to say that without ambiguity, whether a particular object belongs to the collection or not.

By 'distinct' we mean that we do not repeat an object over and over again in a set.

Elements

Each object belonging to a set is called an element (or a member) of the set. Sets are usually denoted by capital letters A, B, C, N, Q, R, S, etc. and the elements by lower case letters. a, b, c, x, y, etc.

The relationship of an object to a set of which it is an element, is called a relation of belonging.

Examples:

(i) Set of natural numbers:

$N = \{1, 2, 3, 4, 5, 6, ...\}$

(ii) Set of integers:

$I = \{0, \pm 1, \pm 2, \pm 3, \pm 4, ...\}$

(iii) ***Set of even integers:*** Any integers of the form 2n, where n is an integer is called an even integer. Thus $0, \pm 2, \pm 4, \pm 6, ...$ are even integers.

(iv) ***Set of odd integers:*** Any integer of the form 2n–1 or 2n+1, where n is an integer is called an odd integer. Thus $\pm 1, \pm 3, \pm 5, ...$ are called odd integers.

(v) ***Set of non-negative integers:*** It contains all positive integers

0, 1, 2, 3, 4, ...

(vii) *Set of all rational numbers*: Any number of the form p/q, where p and q are integers and $q \neq 0$ is called a rational numbers. The set of all rational numbers is denoted by Q.

(viii) *Set of all complex numbers*: Any number of the form $a + ib$.

2.2 TYPE OF SETS

2.2.1 Singleton Set

If a set consists of only one element, it is called a singleton set.

For example:

{1}, {0}, {*a*}, {*b*}, etc.

2.2.2 Finite Set

A set consisting of a natural number of objects, *i.e.*, in which the number of elements is finite, is called finite set.

For example:

A = {5, 7, 9, 11}

and B = {4, 8, 16, 32, 64, 91}

Since A contains 4 elements and B contains 6 elements, so both are finite sets.

2.2.3 Infinite Set

If number of elements in a set is infinite, the set is called infinite set.

For Example: Set of natural numbers.

N = {1, 2, 3, 4, ...} is an infinite set.

2.2.4 Equal Set

Two sets A and B consisting of the same elements are called equal set.

For example:

A = {1, 5, 9}

B = {1, 5, 9}

So here A = B.

2.2.5 Pair Set

A set having two elements is called pair set.

For example:

{1, 2}, {0,3}, (4,9}, {3,1}, etc.

2.2.6 Empty Set

If a set consists of no elements, it is called the empty sector null set or void set and is represented by ϕ.

Here a point is to be noted that ϕ. is a null set but { ϕ.} is singleton set.

Example: Which of the following are valid set definitions?

A = {4, 5, 8}

B = {*a*, *b*, *a*, *c*}

C = {a, b}

D = {1, 2, 3, 4, 5, 6}

S: A is a valid set, containing three elements 4, 5, and 8.

B looks to be equally valid except for the occurring of two a's. We can certainly check for inclusion within the set and this is surely the most important element. Hence we might regard this as valid and equal to {a, b, c}. However, there are problems that original definition of B and remove one of the a's then we apparently have $a \in$ B and $a \in$ B. This conclusion is not allowed since it is inconsistent, hence we shall regard repetition within a set as referring to the same element and its duplication as being an oversight; the removal of duplicates forms the basis of several mathematical arguments later on. C is a valid set, containing 2 elements, a and b.

D is a valid set, containing 6 elements 1, 2, 3, 4, 5 and 6.

2.3 METHODS OF REPRESENTING A SET

The most common method of describing the sets are as follows:

- Roster method or listing method or Tabular method.
- Set Builder method or property method or Rule method.

2.3.1 Roster Method

In this method, a set is described by listing all its elements, separating them by commons and enclosing them within brackets (only brackets).

Example:

(*i*) If A is the set of even natural numbers less than 8.

A = {2, 4, 6}

(*ii*) If B is the set of odd and numbers less than 17.

B = {1, 3, 5, 7, 9, 11, 13, 15}

2.3.2 Set Builder Method

Listing elements of a set is sometimes difficult and sometimes impossible.

In this method, a set is described by means of some property which is shared by all the elements of the set.

Example:

(*i*) If A is the set of all prime numbers then,

A = { x : x is a prime numbers}

(*ii*) If A is the set of all natural numbers between 10 and 1000 then,

A = { x : x N and $10 \leq x \leq 1000$}.

2.4 COMBINATION OF SETS

(*i*) ***Subsets of a given set:*** Let A be a given set. Any set B, each of whose element is also an element of A, is said to be contained in A and is called a subset of A. Subset is represented by ⊂.

Example:

Let A = {*a*, *b*, *c*}, B is a subset of A represented by B ⊂ A then subsets of A are.

{*a*}, {*b*}, {*c*}, {*a*, *b*}, {*b*, *c*}, {*c*, *a*}, {*a*, *b*, *c*} and ϕ.

Example:

Proper subset: If B is subset of A and B ≠ A then B is called proper subset of A. In other words, if each elements of A, which is element of B, then B is called a proper subset of A.

In example (*i*) {*a*} {*b*} {*c*} {*a*, *b*} {*b*, *c*} {*c*, *a*} and ϕ are proper subset (S) of A. and in Example (*ii*) {1}, {2}, {3}, {4}, {1, 2}, {1, 33, 21, 43, {2, 4} {3, 4}, {1, 2, 3}, {1, 2, 4}, {2, 3, 4} and ϕ are proper subset of A.

(*ii*) ***Universal set:*** Any set which is super set of all the sets under consideration is known as the universal set and is denoted by Ω or S.

(*iii*) ***Intersection of sets:*** If P and Q are any two sets then all the elements which are common in P and Q, set of these elements is called intersection of P and Q and is denoted by P ∩ Q.

In Symbols:

$$P \cap Q = \{x | \in P \text{ and } x \in P \text{ and } x \in Q\}$$

Example 1: Let P = {1, 2, 3, 4} and Q = {2, 4, 6, 8}. then

$$P \cap Q = \{2, 4\}$$

Example 2: Let P = {*a*, *b*, *c*, *d*, *e*, *f*} and Q = {*a*, *b*, *c*, *d*, *g*, *h*, *i*, *j*}

$$P \cap Q = \{a, b, c, d\}.$$

(*iv*) ***Union of sets:*** If P and Q are any two sets then all the elements which are either in P or in Q, set of these elements is called union of P and Q and is denoted by P ∪ Q.

In symbols:

$$P \cup Q = \{x | x \in P \text{ or } x \in Q\}$$

Example 1: Let P = {7, 8, 9, 10} and Q = {9, 11, 12, 14}, then

$$P \cup Q = \{7, 8, 9, 10, 11, 12, 14\}$$

Example 2: Let P = {*a*, *b*, *c*, *d*, *e*, *f*} and Q = {*a*, *b*, *c*, *d*, *g*, *h*, *i*, *j*}, then

$$P \cup Q = \{a, b, c, d, e, f, g, h, i, j\}$$

(*v*) ***Disjoint sets:*** Two sets are said to be disjoint. If they have no elements in common. Let A and B are any two sets with no common element.

$$\text{or} A \cap B = \phi$$

i.e. two sets called disjoint if there intersection is a null set.

Example 1: Let A = {1, 2, 3, 4} and B = {5, 6, 7, 8}

$$A \cap B = \phi$$

Since no element is common in set A and B.

Example 2: Let A = {*a*, *b*, *c*} and B = {*e*, *f*, *g*}

So $A \cap B = \phi$

Since no element in A and B is common.

(*vi*) ***Difference of two sets:*** The difference of two sets A and B is the set of all elements which are in A but not in B and is denoted by A – B.

Or symbolically: $A - B = \{x \mid x \in A; x \notin B\}$.

Similarly: The difference of two sets B and is the set of all elements which are in B but not in A and is denoted by B – A.

or

Symbolically:

$$B - A = \{x \mid x \in B; x \notin A\}$$

Example 1: Let A = {1, 2, 3, 4} and B = {2, 4, 6, 8}, then find A – B and B – A

Solution: A – B = {1, 3}

B – A = {6, 8}

Example 2: Let P = {*a*, *b*, *c*, *d*} and Q = {*c*, *d*, *e*, *f*, *g*, *h*}, then find P – Q and Q – P

Solution: P – Q = {*a*, *b*}.

Q – P = {*e*, *f*, *g*, *h*}.

(*vii*) ***Symmetric difference of two sets:*** The symmetric difference of two sets. A and B is the set of the elements which are in the union of (A – B) and (B – A) and is represented by Δ.

So,

$$\Delta_{(A,B)} = (A - B) \cup (B - A)$$

Example 1: Let A = {1, 2, 3, 4, 5, 6} and B = {4, 5, 6, 7, 8, 9, 10} then, find symmetric difference of sets A and B.

Solution: Here A – B = {1, 2, 3}

B – A = {7, 8, 9, 10}

So, $\Delta_{(A,B)} = (A - B) \cup (B - A)$

$= \{1, 2, 3, 7, 8, 9, 10\}$

(*viii*) ***Super set:*** If A and B are two sets and A is the subset of B then B is a super set of A. It can be written as,

$$B \supset A$$

Example 1: A = {1, 2, 3}, B = {1, 2, 3, 4}

Since $A \subset B$

then here, we can say that,

$$B \supset A$$

(*ix*) ***Universal set:*** Any set which is super set of all the sets under consideration is known as the universal set and is denoted by Ω or S.

Example: Let us take some sets

A = {1, 2, 3}, B = {4, 5, 6}, C = {6, 7, 8, 9}

D = {10, 11, 12}, E = {9, 10, 11} F = {6, 7, 9}, then

Universal set,

S = {1, 2, 3, 4, 5, 6 7, 8, 9, 10, 11, or any super set of S}

(*x*) ***Complement of A set:*** Let U be the universal set and $A \subset U$. Then, the complement of A is the set of those elements of U which are not in A. The complement of A is denoted by A^C.

$$A^C = U - A$$

$$= \{x : x \cup \in \text{ and } x \notin A\}$$

$$= \{x : x \notin A\}$$

Complement of a set is set itself.

i.e., $(A^C)^C = A$

Some Symbols

Symbol	Meaning
∈	'belongs to' or 'is a member of'
∉	does not belong to
⊂	is a subset of
⊃	is a super set of
∪	union of sets
∩	intersection of sets
A^C or A′	complement of A
iff	If and only if
: or δ.t. or |	Such as
∀	for all
∃	there exists
⇒	implies

(*xi*) ***Partition of a set:*** A set {A, B, C, ...} of non-empty subset A, B, C, ... of a set S, is called a partition of S if.

(a) $A \cup B \cup C \cup ... S$

(b) The intersection of every pair of distinct subsets is the empty set.

(c) Clearly shows no two of A,B,C. ... have any element in common.

Example: Consider the set S = {1, 3, 5, 7, 9, 11} and its.

Subsets A, B and C such

A = {1, 5, 9}, B = {3, 7}, C = {11}

Clearly (*i*) A, B, C are non empty.

(*ii*) $A \cup B \cup C$ = { 1, 3, 5, 9, 11} = S.

(*iii*) $A \cap B = \phi$, $B \cap C = \phi$ and $C \cap A = \phi$

Hence the set {1, 8, 9}, {3, 7} and {11} is a partition of the set S.

2.5 MULTISETS

Multisets are unordered collections of elements where an element can occur as a member more than once. The notation $\{m_1 \times a_1, m_2 \times a_2, ..., m_r \times a_r\}$ denotes the multiset with element a_1 occurring m_1 times, element a_2 occurring m_2 times, etc. The numbers m_i where i = 1, 2, ..., r, are called the multiplicities of the elements a_i, where i = 1,2, ..., r. Alternatively, a multiset can simply be denoted as a set where repeated elements simply appear multiple times in the representation, such as {*a, a, b, a*}.

Let P and Q be multisets. The union of $P \cup Q$ is a multiset such that the multiplicity of an element in $P \cup Q$ is equal to the *maximum* of the multiplicities of the element in P and in Q. For example,

P = {a, a, a, c, d, d} Q = {a, a, b, c, c} $P \cup Q$ = {a, a, a, b, c, c, d, d}

Note that some definitions of the union of $P \cup Q$ of two multisets P and Q state that the multiplicity of each element in the union is the sum (rather than the maximum) of the multiplicities of the element in P and Q. Such a definition is also useful in some contexts, but for the purposes of this class, we'll simply define the union using the maximum of the multiplicities of the element in the original sets, and define a separate operation (defined later) for the sum. With that said, when dealing with multisets, make sure you know which definition of union you are dealing with!

Now, let set multiset P consist of {EE, EE, EE, ME, MA, MA, CS} — these represent the majors of the personnel needed to complete phase 1 of a project. Multiset Q consists of {EE, ME, ME, MA, CS, CS} - these are the majors of the personnel needed for phase 2 of the project. The multiset $P \cup Q$ gives us the exact number of personnel we need to hire for the project.

The intersection of $P \cap Q$ is a multiset where the multiplicity of an element in $P \cap Q$ is the minimum of the multiplicities of the element in P and in Q. For example,

P = {a, a, a, c, d, d} Q = {a, a, b, c, c} $P \cap Q$ = {a, a, c}

The intersection could be used in the example dealing with the project above to tell us the personnel involved in both phases of the project.

The difference of two multisets P – Q is the multiset such that the multiplicity of an element in the difference P – Q is equal to the multiplicity of the element in P minus the multiplicity of the element in Q if the difference is positive, and 0 otherwise. For example,

P = {a, a, a, b, b, c, d, d, e} Q = {a, a, b, b, b, c, c, d, d, f} P – Q = {a, e}

This would give the personnel that have to be reassigned after the first phase of the project (they are not involved in the second phase).

We define a fourth operation on multisets that does not apply to regular sets. The **sum** of two multisets P + Q is the multiset where the multiplicity of an element in P + Q is equal to the sum of the multiplicities of the element in P and in Q. For example,

P = {a, a, b, c, c} Q = {a, b, b, d} P + Q = {a, a, a, b, b, b. c, c, d}

As an example, think of P as the multiset of all account numbers of transactions made at a bank on a particular day. If Q is the multiset of all account numbers of transactions made the next day, then P + Q is a combined record of account numbers of transactions for the two days. You would want the sum because one account number may have made several transactions in the two days.

Multisets can be useful for a certain class of problems (such as the engineering project example).

Example: P is a multiset that has as its elements the types of computer equipment needed by one department of a university where the multiplicities are the number of items required. Q is the same type of multiset but for a different department.

P = {107 × PCs, 44 × routers, 6 × servers}

Q = {14 × PCs, 6 × routers, 2 × Macs}

Problem 1: What combination of P and Q represents that the university must buy if there is no sharing of equipment?

Solution: Since no equipment is shared, each department needs to have its own version of the equipment. Thus, the combination of equipment needed is :

P + Q = {121 × PC's, 50 × routers, 6 × servers, 2 × Macs}

Problem 2: What combination of P and Q represents, what must be bought if the two departments share?

Solution: If the two departments share, then the number of each item is simply the maximum of the needs of both departments. Thus, the combination of equipment needed is:

$P \cup Q$ = {107 × PC's, 44 × routers, 6 × servers, 2 × Macs}

Problem 3: What combination of P and Q represents the actual equipment that both departments share?

Solution: The combination of equipment shared by the two departments is simply the overlap in the equipment needed by each department. Thus, the combination of equipment shared is:

$P \cap Q$ = {14 × PC's, 6 × routers}.

Solved Example

Example 1: Find the smallest set X such as,

$$X \cup \{1, 2\} = \{1, 2, 3, 5, 9\}$$

Solution: Since

$$X \cup \{1, 2\} = \{1, 2, 3, 5, 9\}$$

So,

Smallest set X contains all the elements except {1, 2}

So,

$$X = \{3, 5, 9\}$$

Example 2: Let A = {1, 2}, B = {2, 4, 6, 8} and C = {3, 4, 5, 6}

Find: $A \cup B$, $B \cap C$, $A - B$, $B - A$, $A \cap (B \cup C)$, $A \cup (B \cap C)$, and $(A \cup B \cup C)$

Solution: Here given,

$$A = \{1, 2, 3\}, B = \{2, 4, 6, 8\} \text{ and } C = \{3, 4, 5, 6\}$$

$$A \cup B = \{1, 2, 3, 4, 6, 8\}$$

$$B \cap C = \{4, 6\}$$

$$A - B = \{1, 3\}$$

$$B - A = \{4, 6, 8\}$$

$$A \cap (B \cup C) = \{1, 2, 3\} \cap \{2, 4, 6, 8\} \cup \{3, 4, 5, 6\}$$

$$= \{1, 2, 3\} \cap \{2, 3, 4, 5, 6, 8\}$$

$$= \{2, 3\}$$

$$A \cup (B \cap C) = \{1, 2, 3\} \cup \{2, 4, 6, 8\} \cap \{3, 4, 5, 6\}$$

$$= \{1, 2, 3\} \cup \{4, 6\}$$

$$= \{1, 2, 3, 4, 6\}$$

$$A \cup B \cup C = \{1, 2, 3\} \cup \{2, 4, 6, 8\} \cup \{3, 4, 5, 6\}$$

$$= \{1, 2, 3, 4, 5, 6, 8\}$$

Example 3: If U = {1, 2, 3, 4, 5, 6, 7, 8, 9}

$$A = \{1, 2, 3, 4\}, B = \{2, 4, 6, 8\} \text{ and } C = \{3, 4, 5, 6\}$$

Find A', $(A \cup B)'$, $(A \cap C)'$ and $(B - C)'$

Solution: Here given

$$U = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}, A = \{1, 2, 3, 4\}, B = \{2, 4, 6, 8\}$$

and

$$C = \{3, 4, 5, 6\}$$

now,

$$A' = U - A$$

$$A' = \{5, 6, 7, 8, 9\}$$

$$A \cup B = \{1, 2, 3, 4, 6, 8\}$$

$$(A \cup B)' = \cup - (A \cup B)$$
$$= \{5, 7, 9\}$$
$$(A \cap C) = \{3, 4\}$$
$$(A \cap C)' = U - (A \cap C)$$
$$= \{1, 2, 5, 6, 7, 8, 9\}$$
$$B - C = \{2, 8\}$$
$$(B - C)' = U - (B - C)$$
$$= \{1, 3, 4, 5, 6, 7, 9\}$$

Example 4: Let $U = R$ the set of all real numbers. If $A = \{ x : x \in R, 0 \leq x \leq 2\}$, $B = \{x : x \in R, 1 \leq x \leq 3\}$ then,

Find A', B', $A \cup B$, $A \cap B$, $A - B$ and $B - A$.

Solution: $A' = R - A = \{x : x \in R \text{ and } x \notin A\}$

$$= \{x : (x \in R \text{ and } x > 2) \text{ or } (x \in R \text{ and } x \leq 0)\}$$
$$= \{x : x \in R \text{ and } x > R\} \cup \{x : x \in R \text{ and } x \leq 0\}$$
$$B' = \{x : x \in R \text{ and } x \leq 1\} \cup \{x : x \in R \text{ and } x > 3\}$$

Similarly:

$$A \cup B = \{x : x \in R \text{ and } 0 \leq x \leq 3\}$$
$$A \cap B = \{x : x \in R \text{ and } 1 \leq x \leq 2\}$$
$$A - B = \{x : x \in R \text{ and } 0 \leq 1\}$$

Example 5: If $A = \{2, 3, 4, 8, 10\}$, $B = \{1, 3, 4, 10, 12\}$ and
$C = \{4, 5, 6, 12, 14, 16\}$ Find $(A \cup B) \cap (A \cup C)$ and $(A \cap B) \cup (A \cap C)$

Solution: Here given $A = \{2, 3, 4, 8, 10\}$, $B = \{1, 3, 4, 10, 12\}$ and
$C = \{4, 5, 6, 12, 14, 16\}$

then,

$$A \cup B = \{1, 2, 3, 4, 8, 10, 12\}$$
$$A \cup C = \{2, 3, 4, 5, 6, 8, 10, 12, 14, 16\}$$
$$(A \cup B) \cap (A \cup C) = \{2, 3, 4, 8, 10, 12\}$$
$$A \cap B = \{3, 4, 10\}$$
$$A \cap C = \{4\}$$
$$(A \cap B) \cup (A \cap C) = \{3, 4, 10\}$$

Example 6: Solve $3x^2 - 12x = 0$ where,

(i) $x \in N$, (ii) $x \in I$, (iii) $D \in S = \{a + ib; b \neq 0, a, b \in R\}$

Solution: We have $3x^2 - 12x = 0$

$$3x(x-4) = 0$$
$$3x(x-4) = 0$$
$$x = 0, 4$$

(i) If $x \in N$ then,

$$x = \{4\}$$

(ii) If $x = I$ then,

$$x = \{0, 4\}$$

(iii) If $x \in S$, then since there is no root of the form $a + ib$ where a and b are constant and $b \neq 0$ the solution set in this case is ϕ.

Example 7: Let $A = \{\theta : 2\cos^2\theta + \sin\theta \leq 2\}$ and $B = \{Q : \pi/2 \leq \theta \leq 3\pi/2\}$ then find $A \cap B$.

Solution: As given sets, B consists of all value of θ in the interval $\pi/2 \leq \theta \leq 3\pi/2$ and set

A consists of all value of θ which satisfy the inequality

$$2\cos^2\theta + \sin\theta \leq 2 \qquad \ldots(i)$$

Hence $A \cap B$ will consist of all those values of θ in the interval $\pi/2 \leq \theta \leq 3\pi/2$ which satisfy the inequality (1),

The inequality (1) is equivalent to the inequality

$$2 - 2\sin^2\theta + \sin\theta \leq 2$$

i.e., $\sin\theta(1 - 2\sin\theta) \leq 0 \ldots > (2)$

The inequality (2) is satisfied by all those values of θ which satisfy $\sin\theta \leq 0$ or $\sin\theta > ½$.

Now the values of θ which lie in the interval $\frac{\pi}{2} \leq \theta \leq 3\pi/2$ and satisfy $\sin\theta \leq 0$ are given by $\frac{\pi}{2} \leq \pi \leq 3\pi/2$. And then values of θ which lie in the interval $\frac{\pi}{2} \leq \theta \leq 3\pi/2$ and satisfy $\sin\theta > \frac{1}{2}$ are given by $\frac{\pi}{2} \leq \theta \leq 5\pi/6$.

Thus the solution set of inequality (1) consists of all values of θ in the intervals $\pi/2 \leq \theta \leq 5\pi/6$ and $\pi/2 \leq \theta \leq 3\pi/2$.

Hence, $A \cap B = \{\theta : \pi/2 \leq \theta \leq 5\pi/6 \text{ or } \pi \leq \theta \leq 3\pi/2\}$

Example 8: If $X = \{4^n - 3n - 1 : n \in N\}$ and $y = \{9(n - 1 : n \in N\}$ prove that $x \subset y$.

Solution: Let $x_n = 4^n - 3n - 1$ ($n \in N$) then,

$$x_1 = 4 - 3 - 1 = 0$$

and for $n > 2$, we have

$$x_n = (1+3)^n - 3n - 1$$

$$= 1 + {}^nc_1\, 3 + {}^nc_2\, 3^2 + \ldots + 3^n - 3n - 1$$

$$= 1 + 3n + {}^nc_2\, 3^2 + \ldots + 3^n - 3n - 1$$

$$= {}^nc^2\, 3^2 + \ldots + 3n = 9\,[{}^nc_2 + \ldots + 3^{n-2}]$$

Thus xn is some five integral of 9 for $n > 2$.

Hence x consists of all positive integral multiple of 9 of the form $9a_n$ where $a_n = nc_2 + nc_3.3 + \ldots + 3^{n-2}$ together with 0.

Also y consists of elements of the form $9\,(n - 1)$ $(n \in N)$ that is, y consists of all positive integral multiple of 9 together with 0.

It follows that $X \subset Y$.

So,

$$x = \{\, 0, 9, 54, 243 \ldots\}$$

and

$$y = \{0, 9, 18, 27, 36, 45, 54 \ldots\}$$

Hence X is a proper subset of Y.

or $X \subset Y$

Example 9: If $9N = \{ax : x \in N\}$. Describe the set $3N \cap 7N$.

Solution: According to the given nation,

$$N = \{3x : x \in N\} = \{\, 3, 6, 9, 12, \ldots\}$$

and

$$7N = \{7x : x \in N\} = \{7, 14, 21, 28, 35, 42, \ldots\}$$

Hence

$$3N \cap 7N = \{21, 42, 63 \ldots\}$$

$$= \{21x : x \in N\} = 21N$$

2.6 LAWS OF OPERATIONS (GENERAL IDENTITIES ON SETS)

If A, B, C are any subset of a universal set $\cup$ then,

1. ***Idempotent Laws:***

   (*i*) $A \cup A = A$

   (*ii*) $A \cap A = A$

2. ***Identity Laws:***

   (*i*) $A \cup \phi = A$

   (*ii*) $A \cap \phi = A$

3. ***Commutative Laws:***

(*i*) $A \cup B = B \cup A$

(*ii*) $A \cap B = B \cap A$

4. ***Associative Laws:***

(*i*) $(A \cap B) \cup C = A \cap (B \cup C)$

(*ii*) $(A \cap B) \cap C = A \cup (B \cap C)$

5. ***Distributive Laws:***

(*i*) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

(*ii*) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

6. ***De-Morgan's Law:***

(*i*) $(A \cup B)' = A' \cap B'$

(*ii*) $(A \cap B)' = A' \cup B'$

i.e., (*i*) Complement of union of two sets is the intersection of their complements.

(*ii*) Complement of intersection of two sets is the union of their complements.

More generally, If $A_1, A_2 \ldots A_n$ are any sets,

$(A_1 \cup A_2 \cup \ldots \cup A_n)' = A_1' \cap A_2' \cap A_3' \ldots \cap A_n'$

$(A_1 \cap A_2 \ldots \cap A_n)' = A_1' \cup A_2' \ldots \cap A_n'$

7. ***Complement Law:***

(*i*) $A \cup A' = S$

(*ii*) $A \cap A' = \phi$

(*iii*) $(A')' = A$

8. ***Law of inclusion:***

(*i*) $A \subset (A \cup B)$ and $B \subset (A \cup B)$

(*ii*) $(A \cap B) \subset A)$ and $(A \cap B) \subset B$

9. ***Law of difference:***

(*i*) $A - \phi = A$

(*ii*) $A - A = \phi$

Proof of Operation on Sets

1. Properties of subset,

(*i*) To prove $A \cup A = A$

Let, $x \in A \cup A \Rightarrow x \in A$ or $x \in A$

$\Rightarrow \quad x \in A$

i.e., $A \cup A \subseteq A$ ...(1)

Conversely,

Let, $x \in A \Rightarrow x \in A$ or $x \in A$

$\Rightarrow \quad x \in A \cup A$

i.e., $A \subseteq A \cup A$ ...(2)

By equations (1) and (2) we get,

$A \cup A = A$

(*ii*) To prove $A \cap A = A$

Here, $x \in A \cap A \quad \Rightarrow x \in A$ and $x \in A$

$\Rightarrow \quad x \in A$

i.e. $A \cap A \subseteq A$ ...(3)

Conversely,

$x \in A \quad \Rightarrow x \in A$ and $x \in A$

$\Rightarrow x \in A \cap A$

$A \subseteq A \quad \in A$ ...(4)

By equations (3) and (4) we get,

$A \cap A = A$

(*iii*) To prove $A \cup B = A$, Iff $B \subseteq A$

$\therefore \quad x \in B \Rightarrow x \in A$ ...(5)

Now $x \in A \cup B \Rightarrow x \in A$ or $x \in B$

$\Rightarrow x \in A$ by equation (5)

i.e., $A \cup B \subseteq A$ ...(6)

Conversely,

$x \in A \quad \Rightarrow x \in A$ or $x \in B$

$\Rightarrow x \in A \cup B$

$A \subseteq A \cup B$ ...(7)

So by equations (6) and (7) we get,

$A \cup B = A$

Now let $A \cup B = A$

i.e., $x \in A$ or $x \in B \Rightarrow x \in A$

$x \in B \Rightarrow x \in A$

Hence, $B \subseteq A$

i.e., $A \cup B = A \Rightarrow B \subseteq A$

or $A \cup B = A$, Iff $B \subseteq A$

(*iv*) To prove $A \cap B = A$, Iff $A \subseteq B$

Let, $A \subseteq B$

$\therefore \quad x \in A \Rightarrow x \in B$ ...(8)

Now $x \in A \cap B \Rightarrow x \in A$ and $x \in B$,

$\Rightarrow x \in A$

i.e., $A \cap B \subseteq A$ ...(9)

and $x \in A \Rightarrow x \in A$ and $x \in B$ by equation (8)

$\Rightarrow x \in A \cap B$

$A \subseteq A \cap B$ ...(10)

By equations (9) and (10) we get,

$A \cap B = A$ ...(11)

Now let $A \cap B = A$

i.e., $x \in A \rightarrow x \in A$ and $x \in B$

$\therefore \quad x \in A \Rightarrow x \in B$

i.e., $A \subseteq B$ ...(12)

By using equations (11) and (12) we get,

$A \cap B = A$ iff $A \subseteq B$

Properties of Intersection Operation on Sets

(*i*) Intersection operation is commutative

i.e., $A \cap B = B \cap A$

Proof: Let $x \in A \cap B$

by the definition of intersection:

$x \in A \cap B$

$\Rightarrow x \in A$ and $x \in B$

$\Rightarrow x \in B$ and $x \in A$

$\Rightarrow x \in B \cap A$

i.e., $A \cap B \subseteq B \cap A$ ...(1)

Conversely

$y \in B \cap A$

By the definition of intersection

$x \in B \cap A$

$\Rightarrow$ $x \in B$ and $x \in A$

$\Rightarrow$ $x \in A$ and $x \in B$

$\Rightarrow$ $x \in A \in B$

i.e., $B \cap A \subseteq A \cap B$ ...(2)

By using equation (1) and (2) we get,

$A \cup B = B \cap A$

(*ii*) Associative law holds in case of intersection of three equal sets.

A, B and C : $(A \cap B) \cap C = A \cap (B \cap C)$

Proof: Let $x \in (A \cap B) \cap C$,

So, $x \in (A \cap B) \cap C$

$\Rightarrow$ $x \in (A \cap B)$ and $x \in C$

$\Rightarrow$ $x \in A$ and $x \in B$ and $x \in C$

$\Rightarrow$ $x \in A$ and $x \in (B \cap C)$

$\Rightarrow$ $x \in A \cap (B \cap C)$.

i.e., $(A \cap B) \cap C \subseteq A \cap (B \cap C)$ ...(1)

Conversely let,

$b \in A \cap (B \cap C)$

$\Rightarrow$ $b \in A$ and $b \in B \cap C$

$\Rightarrow$ $b \in A$ and $b \in B$ and $b \in C$

$\Rightarrow$ $(b \in A$ and $b \in B)$ and $b \in C$

$\Rightarrow$ $b \in (A \cap B)$ and $b \in C$

$\Rightarrow$ $b \in (A \cap B) \cap C$

i.e., $(A \cap B) \cap C \subseteq (A \cap B) \cap C$ ...(2)

By using equations (1) and (2) we get,

$(A \cap B) \cap C = A \cap (B \cap C)$

Properties of Union Operation

(*i*) ***Commutative Law:*** To prove that

$A \cup B = B \cup A$

Proof: Let $x \in A \cup B$

$\therefore \quad x \in A \cup B$

$\Rightarrow \quad x \in A$ or $x \in B$

$\Rightarrow \quad x \in B$ or $x \in A$

$\Rightarrow \quad x \in B \cup A$

$\therefore \quad A \cup B \subseteq B \cup A$ ...(1)

Conversely let, $y \in B \cup A$

But $y \in B \cup A$

$\Rightarrow \quad y \in B$ or $y \in A$

$\Rightarrow \quad y \in A$ or $y \in B$

$\Rightarrow \quad y \in A \cup B$

$\therefore \quad B \cup A \subseteq A \cup B$ ...(2)

By using equation (1) and (2) we have equations:

$A \cup B = B \cap A$

(*ii*) ***Associative Law:*** To prove that,

$A \cup (B \cup C) = (A \cup B)\ C$

Proof: Let, $x \in A \cup (B \cup C)$

Hence, $x \in A \cup B \cup C$

$\Rightarrow \quad x \in A$ or $x \in B$ or x

$\Rightarrow \quad (x \in A$ or $x \in B)$ or $x \in C$

$\Rightarrow \quad x \in (A \cup B)$ or $x \in C$

$\Rightarrow \quad x \in (A \cup B) \cup C$

$\therefore \quad A \cup (B \cup C) \subseteq (A \cup B) \cup C$ ...(1)

Conversely, let $x \in (A \cup B) \cup C$,

Here,

$y \in (A \cup B) \cup C$

$\Rightarrow \quad (x \in A$ or $x \in B)$ or $x \in C$

$\Rightarrow \quad x \in A$ or $x \in B$ or $x \in C$

$\Rightarrow \quad x \in A$ or $(x \in B$ or $x \in C)$

$\Rightarrow \quad x \in A$ or $x \in (B \cup C)$

$\Rightarrow \quad x \in A \cup (B \cup C)$

$\Rightarrow$ (A ∪ B) ∪ C ⊆ A ∪ (B ∪ C) ...(2)

By using equations (1) and (2) we get,

$$A \cup (B \cup C) = (A \cup B) \cup C$$

(iii) ***Distributive laws:*** For three given sets A, B and C to prove that,

(a) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

(b) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

Proof: Let $a \in A \cap (B \cup C)$,

Here $a \in A \cap (B \cup C)$

$\Rightarrow$ $a \in A$ and $a \in B \cup C$

$\Rightarrow$ $a \in A$ and ($a \in B$ or $a \in C$)

$\Rightarrow$ ($a \in A$ and $x \in B$) or ($a \in A$) or $a \in C$)

$\Rightarrow$ $a \in A \cap B$ or $a \in A \cap C$

$\Rightarrow$ $a \in (A \cap B) \cup (A \cap C)$

Hence,

$$A \cup (B \cup C) \subset (A \cap B) \cup (A \cap C) \quad ...(1)$$

Conversely let $b \in (A \cap B) \cup (A \cap C)$

Here, $b \in (A \cap B) \cup (A \cap C) \Rightarrow b \in A \cap B$ or $b \in A \cap C$

$\Rightarrow$ ($b \in A$ and $b \in B$) or ($b \in A$ and $b \in C$)

$\Rightarrow$ ($b \in A$ and $b \in B$) or ($b \in C$)

$\Rightarrow$ ($b \in A$ and $b \in B \cup C$)

$\Rightarrow$ ($b \in A \cap (B \cup C)$)

$\therefore$ $(A \cap B) \cup (A \cap C) \subseteq A \cap (B \cup C)$ ...(2)

By using equations (1) and (2) we get,

Hence, $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

Number of Elements in the Union of two Or more Sets

Let A, B and C be three finite sets and let n(A), n(B), n(C) respectively denote the number of elements in these sets . Then we see that,

$$n(A \cup B) = n(A) + n(B) - n(A \cap B) \quad ...(1)$$

In case A and B are disjoint sets then, $A \cap B = \phi$ and $n(A \cap B) = n(\phi) = 0$

i.e., for disjoint sets A and B

$$n(A \cup B) = n(A) + n(B)$$