CONTENTS

MATHEMATICS-I (DISCRETE MATHEMATICS)

SYLLABUS

UNIT I

Mathematical Logic – Statements and Notation – Connectives: Negation, Conjunction, Disjunction, Statement Formulae and Truth tables – Logical capabilities of programming languages – Conditional and Biconditional well – Formed Formulae Tautologies – Equivalence of Formulae – Duality law Tautological Implications.

UNIT II

Sets, Relations and Functions: Definition of Sets and Subsets: Intersection, Union and Complements DeMorgan's Law; cardinality; Relations – Equivalence relations etc., Mapping One – one; Onto etc.

UNIT III

Algebraic structures: Semigroups and monoids. Groups: Definitions and examples – subgroups and Homomorphisms – Permutations Groups – Cosets and Lagrange's theorem – Normal subgroups – Rings: Definition, types, Matrix, Manipulations, Determinants: Properties of determinants: Grammer's rule: Determinants to transpose and inverse. Properties – canonical forms of a matrix – Cayley – Hamiltonian Theorem – Characteristic Polynomical – Problems.

UNIT IV

Graph Theory: Definition – Examples of graphs – Walks – Paths – Circuits – Trees and fundamental circuits – Cutest and cut vertices – Matrix representation of graphs.

UNIT V

Colouring, Partitioning and covering – Planar Graphs – Directed Graphs – Chromatic Polynomial – Five Colour Theorem.

UNIT I

LESSON

1

MATHEMATICAL LOGIC

CONTENTS

1.0 AIMS AND OBJECTIVES

After studying this lesson, you will be able to:

- Discuss types of statements
- Understand various connectives like negation, conjunction, etc
- Discuss tautology and truth tables
- Discuss logical equivalence
- Understand equivalence of formulae

1.1 INTRODUCTION

Well-defined formula of Proposition: "A proposition is a statement that is either true or false but not both". For example,

1. "Washington D.C is the capital of the United States of America"
2. Toronto is the capital of Canada
3. $1 + 1 = 2$
4. $2 + 2 = 3$
5. What time is it?
6. Read this carefully
7. $x + 1 = 2$
8. $x + 7 = z$

S → Statements 1 and 3 are true, whereas 2 and 4 are false, so they are propositions. Sentences 7 and 8 are not propositions because they are not statements. Sentences 5 and 6 are not propositions because they are neither true nor false.

1.2 TYPES OF STATEMENTS

1. ***Atomic statement (simple statements):*** A statement which has no logical connective in it is called atomic sentences. For example,
   (a) This is my body (true)
   (b) Delhi is the capital of U.P (False).
2. ***Molecular or compound statement:*** The sentence formed by two simple sentences using a logical connective is called a compound sentences. For example,
   (a) U.P. is in India and Lucknow is the capital of U.P.
   (b) Hari will study he will play.

Example 1: Construct the truth table for

$$\sim (P \wedge q) \Leftrightarrow (\sim P \vee \sim q)$$

P	q	$P \wedge q$	$\sim (P \wedge q)$	$\sim P$	$\sim q$	$\sim P \vee \sim q$	$\sim (P \wedge q) \Leftrightarrow (\sim P \vee \sim q)$
T	T	T	F	F	F	F	F
T	F	F	T	F	T	T	T
F	T	F	T	T	F	T	T
F	F	F	T	T	T	T	T

1.3 LOGICAL CONNECTIVES

1.3.1 Negation Operator

Let P be a proposition. The statement "It is not the case that P" is another proposition, called the negation of P. The negation of P is denoted by ΓP.

Example 2: Find the negation of the proposition "Today is Friday" and express this in simple English.

Solution: The negation is "It is not the case that today is Friday"

This negation can be more simply expressed by "Today is not Friday"

Truth Table

"A truth table displays the relationship between the truth values of propositions".

The truth table for the negation of a proportion:

P	ΓP
T	F
F	T

1.3.2 Conjunction Operation

Let P and q be two propositions. The proposition "P and 'V'" denoted by $P \wedge q$, is the proposition that is true when both P and q are true and is false otherwise and denoted by "$\wedge$".

Example 3: Find the conjunction of the propositions P and q where P is the proposition "Today is Friday" and q is the proposition "It is raining today".

Solution: The conjunction of these Propositions P and q where $P \wedge q$ is the proposition "Today is Friday and it is raining today". This is true on rainy Fridays and is false on any day that is not a Friday and on Fridays when it does not rain.

1.3.3 Disjunction Operation

Let P and q be propositions. The propositions "P or q" denoted by $P \vee q$, is the proposition that is false when P and q are both false and true otherwise the proposition $P \vee q$ is called the disjunction of P and q.

The truth table for the conjunction of two propositions

P	q	$P \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

The truth table for the disjunction of two propositions

P	q	$P \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

1.3.4 Exclusive or Operation

Let P and q be propositions. The exclusive or of P and q, denoted by $P \oplus q$, is the proposition that is true when exactly one of P and q is true and is false otherwise.

The truth table for the exclusive or of two propositions

P	q	$P \oplus q$
T	T	F
T	F	T
F	T	T
F	F	F

1.3.5 Implication Operation

Let P and q be propositions. The implication $P \rightarrow q$ is the proposition that is false, when P is true and q is false and true otherwise. In this implication P is called the hypothesis (or antecedent or premise) and q is called conclusion (or consequence).

The truth table for the Implication $P \rightarrow q$

P	q	$P \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

Some of the common ways of expressing this implication are

* "If P the q",	* "P is sufficient for q"
* "P implies q",	* "q if P"
* " If P, q"	* "q whenever P"
* "P only if"	* "q is necessary for P"

1.3.6 Biconditional Operator

Let P and q be propositions the biconditional $P \rightarrow q$ is the proposition that is true when P and q have the same truth values and is false otherwise.

"P if and only if q"

The truth table for the Biconditional $P \leftrightarrow q$:

P	q	$P \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

1.4 LAWS

1. ***Law of Identity:*** Under this law the symbol used will carry the same sense through out the specified problems.
2. ***Law of the Excluded middle:*** Which express that a statement is either true or false.
3. ***Law of Non-contradiction:*** It expresses that no statement is true and false simultaneously.

1.5 STATEMENT FORMULAE

1.5.1 Contingency and Satisfiable

A statement formula (expression involving prepositional variable) that is neither a tautology nor a contradiction is called a contingency.

If the resulting truth value of a statement formula A $(p_1, p_2, \ldots p_n)$ is true for at least one combination of truth values of the variables $p_1, p_2, \ldots, p_n$ then A is said to be satisfiable.

Example 4: Prove that the sentence 'It is wet or it is not wet' is a tautology.

Solution: Sentence can be symbolized by $p \vee \sim p$ where p symbolizes the sentence 'It is wet'. Its Truth table is given below:

p	~p	$p \vee \sim p$
T	F	T
F	T	T

As column 3 contains T everywhere, therefore the given proposition is tautology. This tautology is called the Law of the Excluded Middle.

Example 5: Prove that the proposition $\sim [p \wedge (\sim p)]$ is a tautology.

Solution: The truth table is given below:

P	~P	$P \wedge (\sim p)$	$\sim [p \wedge (\sim p)]$
T	F	F	T
F	T	F	T

Since last column contains 'T' everywhere therefore the given proposition is a tautology.

Example 6: Show that $((p \vee \sim q) \wedge (\sim p \vee \sim q)) \vee q$ is a tautology be definition

Solution: $((p \vee \sim q) \wedge (\sim p \vee \sim q)) \vee q$

$\Rightarrow$ $((p \wedge \sim p) \vee \sim q) \vee q$ (distributive law)

$\Rightarrow$ $(F \vee \sim q) \vee q$

$\Rightarrow$ $(\sim q) \vee q$ (as $p \wedge \sim p = F$)

$\Rightarrow$ T

Example 7: Prove that $(\sim p \vee q) \wedge (p \wedge \sim q)$ is a contradiction.

Solution: Truth table for the given proposition

p	q	~p	~q	$\sim p \vee q$	$p \wedge \sim q$	$(\sim p \vee q) \wedge (p \wedge \sim q)$
T	T	F	F	T	F	F
T	F	F	T	F	T	F
F	T	T	F	T	F	F
F	F	T	T	T	F	F

Since F appears in every row of the last column, therefore the given proposition is contradiction.

1.5.2 Equivalence of Formulae

Two statements A and B in variable $P_1 ...... P_n$ $(n \geq U)$ are said to be equivalent if they acquire the same truth values for all interpretation, i.e., they have identical truth values.

Tautology: A statement that is true for all possible values of its propositional variable is called a tautology. A statement that is always false is called a contradiction and a statement that can be either true or false depending on the truth values of its propositional variables is called a contingency.

Example 8:

P	~P	$P \vee (\sim P)$	$P \wedge (\sim P)$
T	F	T	F
F	T	T	F

Hence PV $(\sim P)$ is a tautology and $P \wedge (\sim P)$ is a contradiction.

1.6 LOGICAL EQUIVALENCE

Two propositions P(p, q, ...) and Q(p, q, ...) are said to be logically equivalent or simply equivalent or equal if they have identical truth tables and is written as

$P(p, q, \ldots\ldots\ldots) \equiv Q(p, q, \ldots\ldots)$

As shown below truth tables of $\neg (p \wedge q)$ and $\neg p \vee q$ are identical. Hence $\neg (p \wedge q) = \neg p \vee \neg q$

P	q	$p \wedge q$	$\sim (p \wedge q)$	P	q	$\neg p$	$\neg q$	$\neg p \vee \neg q$
T	T	T	F	T	T	F	F	F
T	F	F	T	T	F	F	T	T
F	T	F	T	F	T	T	F	T
F	F	F	T	F	F	T	T	T

$\sim (p \wedge q)$ $\qquad$ $\neg p \vee \neg q$

Also when a conditional $p \Rightarrow q$ & its converse $q \Rightarrow p$ are both ture, the statement p & q are said to be logically equivalent. Consider the following table

P	F	$p \wedge F$	$p \vee F$	T	$p \wedge T$	$p \vee T$
T	F	F	T	T	T	T
T	F	F	T	T	T	T
F	F	F	F	T	F	T
F	F	F	F	T	F	T

$p \wedge F \equiv F \quad p \vee F \equiv p \quad p \wedge T \equiv p \quad p \vee T \equiv T$

Hence T is the identity element for the operation $\wedge$ (conjunction) and F is the identity element for the operation $\vee$ (disjunction).

Duality: Any two formulas A and A* are said to be duals of each other if one can be obtained from the other by replaying $\wedge$ by $\vee$ and $\vee$ by $\wedge$.

Example 9:

(a) The dual of $(P \wedge q) \vee r$ is $(P \wedge q) \wedge r$.

(b) The dual of $(P \vee q) \wedge r$ is $(P \wedge q) \vee r$.

Functionally complete Set: "A set of connectives is called functionally complete if every formula can be expressed in terms of an equivalent formula containing the connectives from this set".

$\wedge, \vee, \sim, \Rightarrow, \Leftrightarrow$

Example 10: Consider $P \Leftrightarrow q$

It is equivalent to $(P \Rightarrow q, \wedge (q \Rightarrow P)$

1.7 PREDICATED LOGIC

Well formed formula of Predicate: The statement "x is greater than >" has two parts the first part the variable x is the subject of the statement the second part the predicate "is greater then >" refers to a property that the subject of the statement can have.

$P(x) \rightarrow$ Proposition function P at q.

Quantifiers: When all the variables in a propositional function are assigned values, the resulting statement has a truth value. However, there is another important way called quantification to create a proposition from a propositional function. There are two types of quantification namely universal quantification and existential quantification.

Many mathematically statement assert that a property is true for all values of a variable in a particular domain, called the universe of discourse. Such a statement is expressed using a universal quantification. The universal quantification of a propositional function is the proposition that asserts that P(x) is true for all values of x in the universe of discourse. The universe of discourse specifies the possible values of the variable x.

Arguments: An argument (denoted by the symbol $\vdash$ which is called trunstile) is a sequence of propositions that purport to imply another proposition.

The sequence of propositions serving as evidence will be called, the premises, and the proposition inferred will be called the conclusion.

An argument is valid if and only if whenever the conjunction of the premises is true, the conclusion is also true. If we let p_1, p_2, p_3 be the premises and p_4 the conclusion then argument $p_1, p_2, p_3, \vdash p_4$ will be valid if and only if whenever $p_1 \wedge p_2 \wedge p_3$ is true, p_4 is also. We can reduce this to the conditional $\Rightarrow$ as follows:

Def^n: If $p_1, p_2, \ldots, p_n$ are premises and p is a conclusion then the argument $p_1, p_2, \ldots, p_n \vdash p$ is valid if and only if $p_1 \wedge p_2 \wedge \ldots p_n \Rightarrow p$ is true for all combinations of truth values of $p_1 p_2 \ldots p_n$ and p.

In other word in order to decide whether an argument is valid, use the conjunction of evidences as the antecedent of conditional of which the conclusion of the argument is the consequent and see whether or not a tautology results.

If $p_1 \wedge p_2 \wedge \ldots \wedge p_n \Rightarrow p$ is not a tautology then the argument $p_1 \ldots p_n \vdash$ is invalid.

Example 11: "If the labour market is perfect then the wages all persons in a particular employment will be equal. But it is always the case that wages for such persons are not equal therefore the labour market is not perfect". Test the validity of this argument.

Solution: In the given case let

p_1 : "The labour market is perfect"

p_2 : "Wages of all persons in a particular employment will be equal"

$\sim p_2$: Wages for such persons are not equal.

$\sim p_1$: The labour market is not perfect.

The premise are $p_1 \Rightarrow p_2$, $\sim p_2$ and the conclusion is $\sim p_1$

The argument $p_1 \Rightarrow p_2 \sim p_2 \vdash \sim p_1$ is valid if and only if $(p_1 \Rightarrow p_2) \Rightarrow \sim p_2 \Rightarrow \sim p_1$ is a tautology.

We construct the truth tables as below:

p_1	p_2	$\sim p_1$	$\sim p_2$	$p_1 \Rightarrow p_2$	$p_1 \Rightarrow p_2 \wedge \sim p_2$	$p_1 \Rightarrow p_2 \wedge \sim p_2 \Rightarrow \sim p_1$
T	T	F	F	T	F	T
T	F	F	T	F	F	T
F	T	T	F	T	F	T
F	F	T	T	T	T	T

It follows that $p_1 \Rightarrow p_2 \wedge \sim p_2 \Rightarrow \sim p_1$ is a tautology

Hence the argument is valid.

Example 12: Test the validity of the following argument. "If Ashok wins then Ram will be happy. If Kamal wins Raju will be happy. Either Ashok will win or Kamal will win. However if Ashok wins, Raju will not be happy and if Kamal wins Ram will not be happy. So Ram will be happy if and only if Raju is not happy."

Solution: Here let

p_1 : Ashok wins

p_2 : Ram is happy

p_3 : Kamal wins

p_4 : Raju is happy

The premises are $p_5 : p_1 \Rightarrow p_2$, $p_6 : p_3 \rightarrow p_4$, $p_7 : p_1 \vee p_2$

$p_8 : p_1 \Rightarrow \sim p_4$

$p_9 : p_3 \Rightarrow \sim p_2$

The conclusion is $p_1 : p_2 \Leftrightarrow \sim p_4$

The above argument is valid if $p_1 : p_2 \Leftrightarrow \sim p_4$ is a tautology so we construct the truth table.

1	2	3	4	5	6	7	8	9	10	11	12	13
p_1	p_2	p_3	p_4	$p_1 \Rightarrow p_2$	$p_3 \Rightarrow p_4$	$p_1 \vee p_3$	$\sim p_4$ or	$p_1 \Rightarrow \sim p_4$	$p_3 \Rightarrow \sim p_2$	$p_2 \Rightarrow \sim p_4$	$p_1 \Rightarrow p_2 \wedge p_3 \Rightarrow p_4 \wedge (p_1 \wedge p_3) \Rightarrow p_1 \Rightarrow \sim p_4 \wedge p_3 \Rightarrow p_2$	(12)⇒ ⇒(13)
T	T	T	T	T	T	F	F	F	F	F	F	T
T	T	T	T	T	T	F	F	T	F	F	F	T
T	F	T	F	F	F	T	T	T	T	F	F	T
T	F	F	F	F	T	T	T	T	T	F	F	T
F	T	T	T	T	T	T	F	T	F	F	F	T
F	T	F	T	T	T	F	F	T	T	F	F	T
F	F	T	F	T	F	T	F	T	T	F	F	T
F	F	F	F	T	F	F	T	T	T	F	F	T

Since the given statement is a tautology. Hence the argument is valid.

1.8 ALGEBRA OF PROPOSITIONS

We can easily verify by truth tables that every proposition satisfy the following laws. If P and Q are propositions and Fare tautology and contradiction respectively.

1.8.1 Associative Laws

(a) $(P \vee Q) \vee R = P \vee (Q \vee R)$

(b) $(P \wedge Q) \wedge R = P \wedge (Q \vee R)$

1.8.2 Commutative Laws

(a) $P \vee Q = Q \wedge P$

(b) $P \wedge Q = Q \wedge P$

1.8.3 Distributive Laws

(a) $P \vee (Q \wedge R) = (P \vee Q) \wedge (P \vee R)$

(b) $P \wedge (Q \vee R) = (P \wedge Q) \vee (P \wedge R)$

1.8.4 DeMorgan's Laws

(a) $\sim (P \vee Q) \equiv \sim P \wedge \sim Q$

(b) $\sim (P \wedge Q) \equiv \sim P \vee \sim Q$

(c) $\sim (P \Rightarrow Q) \equiv \sim P \wedge \sim Q$

(d) $\sim (P \Leftrightarrow Q) \equiv P \Leftrightarrow \sim Q \ \sim P \Leftrightarrow Q$

1.8.5 Identity Laws

(a) $P \vee F = P$

(b) $P \vee \sim P \equiv F$

(c) $\sim \sim P \equiv P$

(d) $\sim T \equiv F$

(e) $\sim F \equiv T$

1.8.6 Idempotent Laws

(a) $P \vee P \equiv P$

(b) $P \wedge P \equiv P$

1.8.7 Compliment Laws

(a) $P \vee \sim P = T$

(b) $P \wedge \sim P \equiv F$

(c) $\sim \sim P \equiv P$

(d) $\sim T \equiv F$

(e) $\sim F \equiv T$

1.9 EXAMPLES ON TRUTH TABLES

Example 13: Prove that $p \vee (q \wedge r)$ and $(p \vee q) \wedge (p \vee r)$ have the same truth table i.e. they are equivalent.

Solution:

1	2	3	4	5	6	7	8
p	q	r	$q \wedge r$	$p \vee q$	$p \vee r$	$p \vee (q \wedge r)$	$(p \vee q) \wedge (q \vee r)$
T	T	T	T	T	T	T	T
T	T	F	F	T	T	T	T
T	F	T	F	T	T	T	T
F	T	T	T	T	T	T	T
F	F	T	F	F	T	F	F
F	T	F	F	T	F	F	F
T	F	F	F	T	T	T	T
F	F	F	F	F	F	F	F

Since the truth values of the columns (7) and (8) are the same for all values of the component of sentences. Hence the given functions are equivalent.

Example 14: Using, truth table prove the DeMorgan's laws

(i) $\sim (p \wedge q) \equiv \sim p \vee \sim q$

(ii) $\sim (p \vee q) \equiv \sim p \wedge \sim q$

Solution: Consider the truth table

1	2	3	4	5	6	7
p	q	$p \wedge q$	$-p$	$-q$	$\sim (p \wedge q)$	$(\sim p \vee \sim q)$
T	T	T	F	F	F	F
T	F	F	F	T	T	T
F	T	F	T	F	T	T
F	F	F	T	T	T	F

Since the truth values of the columns (6) and (7) are same law both functions are equivalent.

Similarly we can prove (U) result.

Example 15: Prove that $(p \Rightarrow q) \vee r \equiv (p \vee r) \Rightarrow (q \vee r)$

Solution: Consider the truth table

1	2	3	4	5	6	7	8
p	q	r	$p \Rightarrow q$	$(p \Rightarrow q) \vee r$	$p \vee r$	$q \vee r$	$(p \vee r) \Rightarrow (q \vee r)$
T	T	T	T	T	T	T	T
T	T	F	T	T	T	T	T
T	F	T	F	T	T	T	T
F	T	T	T	T	T	T	T
T	F	F	F	F	T	F	F
F	F	T	T	T	F	T	T
F	T	F	T	T	T	T	T
F	F	F	T	T	F	F	T

Since the truth values of the columns (5) and (8) are same hence the given sentences are logically equivalent.

Example 16: Simplify the following propositions:

(a) $P \vee (P \wedge Q)$

(b) $(P \vee Q) \wedge \sim P$

(c) $\sim (P \vee Q) \vee (\sim P \wedge Q)$

Solution:

(a) $P \vee (P \wedge Q)$

$\equiv (P \wedge T) \vee (P \wedge Q)$ (since $P \wedge T = P$)

$\equiv P \wedge (T \vee Q)$

$\equiv P$ (since $T \vee Q = T$)

(b) $(P \vee Q) \wedge \sim P$

$\equiv \sim P \wedge (P \vee Q)$ (Commutative Law)

$\equiv (\sim P \wedge P) \vee (\sim P \vee Q)$ (Distributive Law)

$\equiv F \vee (\sim P \vee Q)$ (Complement Law)

$\equiv (\sim P \vee Q)$ (Identity Law)

$= \quad \sim P \vee Q$

(c) $\sim (P \vee Q) \vee (\sim P \wedge Q)$

$\Rightarrow (\sim P \wedge \sim Q) \vee (\sim P \wedge Q)$

$\equiv \sim P(\sim Q \vee Q)$ (By distributive law)

$\equiv \sim P \wedge T\,(\sim Q \vee Q = T)$

$\equiv \sim P\,(P \wedge T = P)$

Example 17: Construct truth table for the following functions and check whether it is a tautology or contradiction.

(a) $[(p \wedge q) \vee (q \wedge r) \vee (r \wedge p)] \Leftrightarrow [(p \vee q) \wedge (q \vee r) \wedge (r \vee p)]$

Solution: Truth table corresponding to the given function is given below:

p	q	r	$p \wedge q$	$q \wedge r$	$r \wedge p$	$p \vee q$	$q \vee r$	$r \vee p$	$(p \wedge q) \vee (q \wedge r) \vee (r \vee p)$	$(p \vee q) \wedge (q \vee r) \wedge (r \vee p)$
T	T	T	T	T	T	T	T	T	T	T
T	T	F	T	F	F	T	T	T	T	T
T	F	T	F	F	T	T	T	T	T	T
F	T	T	F	T	F	T	T	T	T	T
T	F	F	F	F	F	T	F	T	F	F
F	T	F	F	F	F	T	T	F	F	F
F	F	T	F	F	F	F	T	T	F	F
F	F	F	F	F	F	F	F	F	F	F

Since the last two columns of the above table are the same. Hence given function represents a tautology.

(b) $(p \Rightarrow q) \Leftrightarrow (\sim p \vee q)$

p	q	$\sim p$	$p \Rightarrow q$	$\sim p \vee q$	$p \Rightarrow q \Leftrightarrow (\sim p \vee q)$
T	T	F	T	T	T
T T	F	F	F	F	T
F F	T	T	T	T	T
F	F	T	T	T	T

Since the last column contains all truth values T. Hence given function represents a tautology.

(c) $(p \wedge q) \Rightarrow p] \Rightarrow [q \wedge \sim q]$

p	q	$\sim q$	$p \wedge q$	$(p \wedge q) \Rightarrow p$	$q \wedge \sim q$	$(p \wedge q) \Rightarrow p \Rightarrow [q \wedge \sim q]$
T	T	F	T	T	F	F
T	F	T	F	T	F	F
F	T	F	F	T	F	F
F	F	T	F	T	F	F

Since the last column contains all truth values F. Hence the given function represents a contradiction.

(d) $[(p \wedge q) \Rightarrow r] \Leftrightarrow [(p \Rightarrow r) \vee (q \Rightarrow r)]$

Solution:

p	q	r	$p \wedge q$	$(p \Rightarrow q)$	$q \Rightarrow r$	$(p \wedge q) \Rightarrow r$	$(p \Rightarrow q) \vee (q \Rightarrow r)$	(d)
T	T	T	T	T	T	T	T	T
T	T	F	T	T	F	F	T	F
T	F	T	F	F	T	T	T	T
F	T	T	F	T	T	T	T	T
T	F	F	F	F	T	T	T	T
F	T	F	F	T	F	T	T	T
F	F	T	F	T	T	T	T	T
P	F	F	F	T	T	T	T	T

Since the last column of above table does not contain all truth values T, so, it is not a tautology.

Example 18: Establish the equivalence using truth tables.

(a) $(p \Rightarrow q) \vee (p \Rightarrow r) \equiv p \Rightarrow (q \vee r)$

Solution: Truth table for the given statement is

p	q	r	$q \vee r$	$p \Rightarrow q$	$p \Rightarrow r$	$(p \Rightarrow q) \vee (p \Rightarrow r)$	$p \Rightarrow (q \vee r)$
T	T	T	T	T	T	T	T
T	T	F	T	T	F	T	T
T	F	T	T	F	T	T	T
F	T	T	T	T	T	T	T
T	F	F	F	F	F	F	F
F	T	F	T	T	T	T	T
F	F	T	T	T	T	T	T
F	F	F	F	T	T	T	T

Since the last two columns of the above table have the same truth values. Hence the given statements are logically equivalent.

(b) $(p \Rightarrow q) \Rightarrow (p \wedge q) = (\sim p \Rightarrow q) \wedge (q \Rightarrow p)$

Solution: We construct the truth table

p	q	$\sim p$	$p \wedge q$	$p \wedge q$	$\sim p \Rightarrow q$	$q \Rightarrow p$	$(p \wedge q) \Rightarrow (p \wedge q)$	$(\sim p \Rightarrow q) \wedge (q \Rightarrow p)$
T	T	F	T	T	T	T	T	T
T	F	F	F	F	T	T	T	T
F	T	T	T	F	T	F	F	F
F	F	T	T	F	F	T	F	F

Since the last two columns have the same truth values. Hence given statements are logically equivalent.

(c) $(p \vee q)\ (p \wedge q) \equiv p$

Solution: *Truth table is*

p	q	$p \vee q$	$p \wedge q$	$(p \vee q) \vee (p \wedge q)$
T	T	T	T	T
T	F	T	F	T
F	T	T	F	T
F	F	F	F	F

Since the first and last columns of the given statement do not contain same truth values. Hence given statements are not logically equivalent.

(d) $p \Rightarrow (q \wedge r) \equiv (p \Rightarrow q) \wedge (p \Rightarrow r)$

Solution: Truth Tables given by

p	q	r	q	$p \Rightarrow q$	$p \Rightarrow r$	$p \Rightarrow (q \wedge r)$	$(p \Rightarrow q) \wedge (p \Rightarrow r)$
T	T	T	T	T	T	T	T
T	T	F	F	T	F	F	F
T	F	T	F	F	T	F	F
F	T	T	T	T	T	T	T
T	F	F	F	F	F	F	F
F	T	F	F	T	T	T	T
F	F	T	F	T	T	T	T
F	F	F	F	T	T	T	T

Since the last two columns of the above table have the same truth values hence the given statements are logically equivalent.

Example 19: Establish the equivalence analytically and write dual also of the given statement

$$\sim p \wedge (\sim q \wedge r) \vee (q \wedge r) \vee (p \wedge r) \equiv r.$$

Solution: Consider L.H.S.

$$\sim p \wedge (\sim q \wedge r) \vee (q \wedge r) \vee (p \wedge r)$$

$$\equiv [(\sim p \wedge \sim q) \wedge r) \vee [[(q \wedge r) \vee p] \vee [(q \wedge r) \vee r]$$

$$\equiv [(\sim p \wedge \sim q) \wedge r) \vee [\{(q \wedge r) \vee p\} \vee r]$$